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Attraction basins in discretized maps

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Abstract. In this note we consider maps which are defined on continuous space whose large time behaviour displays a strange attractor. We are interested in the properties of the discrete maps that are obtained from these continuous ones by discretizing the space. Such systems behave as disordered dynamical systems. The strange attractor breaks down in many (sometimes one) periodic attractors. We study here the statistical properties of such attractors. Generalizing previous conjectures we propose that the distribution of the attraction basins' sizes is the same as in the random map problem. This result is shown to be in good agreement with numerical experiments.

1. Introduction

It is well known that there are many maps defined on a continuous space whose asymptotic evolution takes place on a strange attractor, i.e. a set whose fractal dimension is in general not equal to the dimension of the embedding space and whose dynamics shows sensitive dependence on initial conditions.

More precisely we have in mind the following situation: x is a vector in R^D; there is a transformation $f(x)$ which is defined on a closed set $\mathcal{D}$ and brings $\mathcal{D}$ in itself. A trajectory of this system is a sequence x_n such that

$$x_{n+1} = f(x_n). \tag{1}$$

According to the Ruelle's definition [1], an attractor is a set $\mathcal{A}$ contained in a D-dimensional neighbourhood $\mathcal{D}$ which is invariant under the dynamics described by (1). It must be *attracting*, i.e. x_n stays as close as one wants to $\mathcal{A}$ for an initial condition chosen in $\mathcal{D}$ and n large enough, and *undecomposable*, i.e. $\mathcal{A}$ cannot be decomposed into two non-trivial invariant parts. If the dynamics shows sensitive dependence on initial conditions $\mathcal{A}$ is called a *strange attractor*. In general, they have non-trivial fractal dimensions, defined for instance through the operation of box counting. For generic initial conditions the trajectory asymptotically stays on the attractor $\mathcal{A}$ and the dynamics on $\mathcal{A}$ is chaotic, in the sense that there is a large enough number of positive Lyapunov exponents. The map is expansive when restricted to the attractor and is contracting in the perpendicular directions, in such a way that the distance from the attractor decreases very fast with the time n. A precise mathematical treatment of the problem can be done in the case of Anosov systems [3].

Two of the simplest and best studied examples are the logistic map and the Henon map, which are defined respectively in one- and two-dimensional space. The maps $f(x)$ are respectively given by

$$x_{n+1} = 1 - ax_n^2 \tag{2}$$

and

$$x_{n+1} = 1 - ax_n^2 + y_n \qquad y_{n+1} = bx_n. \tag{3}$$

This situation is extremely well studied in the literature (see, for example, [2] for a review). In this note we address a slightly modified problem. We discretize the space introducing a lattice space ϵ. This can be done in several ways. In a one-dimensional problem, we associate to each integer number i the real number $X(i) \equiv \epsilon i$, and to each real number x the integer number $i \equiv I(x)$, where $I(x)$ can be chosen, for example, as the integer number which minimizes $x - X(i)$ (in other words, $I(x)$ is the integer which is nearest to x/ϵ).

We now define a mapping on integers by the relation

$$i_{n+1} = I(f(X(i_n))) \equiv f_\epsilon(i_n). \tag{4}$$

Otherwise stated, we start from an integer, we find the corresponding point in the space, we apply the map f to it and we convert the result into an integer again. In this way we obtain a mapping among integers which, barring pathologies, for sufficiently small ϵ brings the integers corresponding to the domain $\mathcal{D}$ into themselves.

What happens for small ϵ? At first we could think that the perturbation that we introduced is small. However, there are new features which are present only in this case. A set containing a finite number of elements, $O(\epsilon^{-D})$, is carried into itself and therefore all orbits are periodic. Moreover, the strange attractor may break into more than one periodic orbit. In this case, if there are many periodic orbits, labelled by an index α, we can define the period L_α as the length of orbit α and the weight w_α as the probability to pick up at random an initial configuration which ends up on the periodic orbit α. For each value of ϵ the w's and the L's are computable numbers, which depend on ϵ in a rather complicated way. We are interested in their statistical properties, in the limit $\epsilon \to 0$.

This situation attracted much interest in the literature in the past decade, starting from the numerical studies by Rannou [4], Levy [5] and Beck and Roepstorf [6]. The problem is not academic: what we have just described is what happens during a computer simulation of a deterministic dynamical system, due to the round off introduced by floating point operations. Indeed real numbers are represented with a finite accuracy, the different operations among these numbers are done with higher accuracy and the result is rounded at the end. The question discussed in this paper can be rephrased as a study of the fate of the strange attractors when rounded arithmetic is used.

Grebogi *et al* [7] and Beck [8] showed that the typical cycle lengths scale as a power law of the inverse lattice spacing, the exponent being the so-called 'correlation dimension' of the attractor (see below). Two related arguments were given to justify this conclusion. Both are based on the analogy with a dynamical system where the dynamic rules, which are obeyed deterministically, are chosen at random at the beginning and kept fixed: the random map model [9, 10]. In section 2 we will summarize the definition and the main properties of the random map model and of the disordered dynamical systems that generalize it. In section 3, which is the central one, we will deal with the relationship between discretized maps and disordered dynamical systems. The probabilistic scheme that we introduced in [11] for the study of the dynamics of genetic regulatory systems, based on the closing probabilities, will play the role of a bridge between the two kinds of models. This scheme allows us to compute directly the distribution of the weights of attraction basins, which turns out to be the same as in the case of the random map model [10]. Numerical simulations of the two maps illustrated above confirm this conclusion, when parameter values are chosen so that the dynamics is chaotic, and show a different behaviour at the transition between chaotic and regular motion. They are presented in section 4. They also throw some light on the analogy between discretized maps and disordered dynamical systems.

Let us come back to this analogy. This is related to an issue in which we are very interested: the problem of the relationship between systems which have a fixed evolution law (in short, deterministic systems) and systems in which the evolution laws (e.g. the Hamiltonian) are random (random systems), whose best-studied examples are spin glasses [12]. In this case the words deterministic and random do not refer to the nature of the equations of motion†, but to the choice of these equations.

A typical example is the problem of the distribution of the energy levels in a quantum Hamiltonian system which is chaotic at the classical level. It is widely believed, and serious progress in this direction has been recently made, that the distribution of the levels is the same as in the random matrix theory, in the limit in which the size N of the matrix goes to infinity [13].

A similar issue is the relation between some models of statistical mechanics without disorder (e.g. minimum autocorrelation sequences) and the properties of spin-glass-like systems where the Hamiltonian is chosen at random [14].

In the same spirit an open and crucial issue is whether real glasses (which have a given Hamiltonian, e.g. a Lennard-Jones potential) are in the same universality class (as far as the glassy transition is concerned) of disordered systems in which the Hamiltonian is chosen at random [15].

The investigation of the relationship between the discretization of a continuous map and a random map gives information on another side of this problem.

2. On disordered dynamical systems

The random map is one of the best known examples of disordered dynamical systems. Its phase space is constituted by a finite set of points with no underlying metric structure: $\Omega = \{1, \dots, M\}$. On this space the dynamics is deterministic, but dynamic rules are chosen at random at the beginning in the following way: we define a map f_η of Ω into itself (the subscript η labels the realization of the dynamic rules) by extracting at random for each point i its successor, $f_\eta(i)$. Successors of different points are independent random variables. When they have all been extracted, the dynamic rules

$$i_{n+1} = f_\eta(i_n) \tag{5}$$

are followed deterministically. In this way, we deal with the ensemble of all the M^M maps that can be built with M points, and we are interested in their statistical behaviour in the large-M limit.

With the usual definition of the RM, all points of phase space have the same probability, $1/M$, of being extracted as successors of a given point. This definition is generalized easily by letting the probability p_j that j is extracted be non-uniform, but again independent on the starting point i: see, e.g. [8].

A more interesting generalization consists in considering correlated dynamical rules, i.e. $f_\eta(i)$ and $f_\eta(j)$ are correlated random variables, and may depend in principle on the starting points i and j. We will call such systems where the dynamics is deterministic but dynamical rules are extracted at random and kept fixed during the evolution *disordered dynamical systems*. Outstanding examples are met in biological modelling: for instance, the

† In the case of a map or of a Hamiltonian system, these are always deterministic.

random boolean networks proposed by Kauffman as a model of genetic regulatory systems [16, 11], or attractors neural networks [17].

In [11] we defined a general scheme to study the statistical properties of the attractors in disordered dynamical systems, based on the so-called closing probabilities $\pi_M(t, t')$: they represent the conditional probabilities that a trajectory chosen at random in a random realization of the dynamics visits the same configuration at times t and t', given that no configuration was repeated before the closing time t'. This means that the trajectory, after a transient time t, has entered a periodic orbit of length $l = t' - t$. In terms of the closing probabilities, the probability to find such a trajectory is easily computed. We have to compute the probability $F_M(t)$ that the trajectory was not closed before time $t' = t + l$. This obviously follows the equation $F_M(t + 1) = F_M(t)(1 - \sum_{t'=0}^{t-1} \pi_M(t', t))$, whence, introducing a continuous time variable, it follows that

$$F_M(t) = \exp\left(- \int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t'' \pi_M(t', t'')\right). \tag{6}$$

(To have a slightly simpler formula, we are supposing that the typical closing times are long. This is verified under the hypothesis that we will discuss in this paper, since the time-scale grows as a power law of M.) Thus we can express in terms of closing probabilities the probability to find a trajectory which, after a transient time t, ends up on a cycle of length l:

$$P\{T = t, L = l\} = \pi_M(t, t + l) \exp\left(- \int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t'' \, \pi_M(t', t'')\right). \tag{7}$$

If, as it is the case for the RM, the closing probabilities converge asymptotically to a value π_M^* independent of both t and t', the above formula can be put into the form

$$P\{T = t, L = l\} = \frac{1}{\tau_M^2} \exp\left(-\frac{(t + l)^2}{2\tau_M^2}\right) \tag{8}$$

where $\tau_M = {\pi_M^*}^{-1/2}$ is the typical time-scale of the problem, in the sense that the random variables t/τ and l/τ have a well-defined density of probability even in the limit where τ goes to infinity.

All the dependence on system size is thus contained in the factor π_M^*, which is expected to decrease as a power law of M under the hypothesis that we will discuss below. For a uniform RM it holds $\pi_M^* = 1/M$, and consequently the typical time-scale of the dynamical systems grows as $\sqrt{M}$, while for a non-uniform RM the following holds:

$$\pi_M^* = \sum_{i=1}^{M} p_i^2 \tag{9}$$

and the typical time-scale is in general shorter than in the uniform case (the result for the non-uniform RM was obtained in [7] and in [8]).

Once this probability is known, it is possible to obtain the average number of attractors existing in a given realization of the dynamical rules: the average number of attractors of length l is obviously given by

$$n_a(l) = \frac{M}{l} P\{L = l, T = 0\}. \tag{10}$$

Now we cannot use formula (8), since the closing probability has not yet reached its asymptotic value for $T = 0$. Instead, we have to calculate the closing probability $\pi_M(0, l)$. In the cases where we performed an approximate computation of this quantity [11, 19], we saw that, under the same hypothesis which implies an asymptotic value of the closing probability, it holds that $\pi_M(0, l) \approx 1/M$ for large enough l, just as in a uniform random map. Logarithmic corrections to this behaviour can be found, for instance in the 'critical' case where π^*_M decreases only logarithmically with system size. Ignoring this possibility, we find, for long cycles,

$$n_a(l) \approx \frac{1}{l} \exp\left(-\frac{l^2}{2\tau_M^2}\right) \tag{11}$$

and, summing over l, we get for the leading term in M

$$\sum_l n_\alpha(l) \approx \log \tau_M \tag{12}$$

so that, for a uniform RM, the average number of cycles is proportional to the logarithm of the number of configurations.

The other quantity which characterizes the statistical properties of the attractors is the distribution of attraction basin weights. This was analytically computed by Derrida and Flyvbjerg for the case of the uniform RM [10]. The computation need not to be modified for the case of a generic disordered dynamical system where the closing probabilities have a constant asymptotic value, and does not depend on the value of π^* (thus it is also independent of system size), in the large-size limit. For the sake of completeness, let us report briefly the main steps in the calculation of [10], starting from the definition of the attraction basin weights.

Let α be the label of the periodic orbit Γ_α; then the weight w_α denotes the probability that a trajectory starting form a random configuration ultimately reaches Γ_α. The weights satisfy the normalization condition:

$$\sum_\alpha w_\alpha = 1. \tag{13}$$

Following [10], it is convenient to consider the 'moments' of the weight distribution $\overline{Y_p}$, defined as

$$\overline{Y_p} = \sum_\alpha \overline{w_\alpha^p} \tag{14}$$

where the overline denotes an average over the realizations of the dynamical rules.

These quantities may be computed if one notes that Y_p represents the probability that p randomly chosen initial conditions end up on the same periodic orbit. For example, let us compute $\overline{Y_2}$ [10]: we extract two trajectories, and we compute the probability that the first one touches $t_1 = x\tau$ different configurations before closing and that the second one touches $t_2 = y\tau$ different configurations before closing on one of the t_1 configurations of the second trajectory. We assume that the closing probability of the second trajectory on the first one is equal to the closing probability of a trajectory on itself. In the large-τ limit and if τ is much larger than the time needed for $\pi(t, t')$ to attain its stationary value, we obtain for the above probability $x^2/\tau^2 \exp(-(x+y)^2/2)$. Thus, summing over t_1 and t_2, we finally get

$$\overline{Y_2} = \int_0^\infty \mathrm{d}x \int_0^\infty \mathrm{d}y \, \exp\left(-\frac{(x+y)^2}{2}\right) = \frac{2}{3}. \tag{15}$$

We stress that this formula does not depend on the value of τ, thus it does not depend on the particular kind of generalized random map considered, provided the closing probability has a stationary value [11].

It is possible to generalize this procedure to get the generic $\overline{Y_p}$:

$$\overline{Y_P} = \frac{4^{P-1}[(P-1)!]^2}{(2P-1)!} \tag{16}$$

(note that these are independent of τ).

This relation can be inverted to obtain the density of the number of weights: the function $f(w)$ (not to be confused with a map!), whose meaning is that $f(w)\,\mathrm{d}w$ represents the average number of attractors of weight comprised between w and $w+\mathrm{d}w$. The following holds [10]:

$$f(w) = \tfrac{1}{2}w^{-1}(1-w)^{-1/2}. \tag{17}$$

The integral $\int \mathrm{d}w\, f(w)$, if finite, would be the average value of the total number of periodic attractors. Its logarithmic divergence is a hint that the total number of periodic attractors diverges as $\ln(M)$ when $M \to \infty$.

This distribution of the w's is very similar to the one found in spin glasses, where w_α is defined as the weight of the pure thermodynamical state α in the Boltzmann state. There is a simple relation between the two distributions, which can be found in [18].

Not only can the average value of the w's be computed, but also the fluctuations from system to system can be predicted. For example we can define for each system the quantity

$$Y \equiv Y_2 = \sum_\alpha (w_\alpha)^2 \tag{18}$$

whose average value is $\frac{2}{3}$.

A detailed computation shows that

$$\overline{Y^2} - \overline{Y}^2 = \tfrac{52}{105} - \tfrac{4}{9} = 0.0508 \tag{19}$$

so that Y fluctuates from sample to sample also in the infinite-size limit.

These are the predictions for the generalized RM which we will compare with the behaviour of the discretized maps.

3. Discretized maps as disordered dynamical systems

We now consider the dynamical behaviour of a discretized chaotic map. If the domain $\mathcal{D}$ which is mapped into itself has a volume V, the mapping f_ϵ acts on a discrete space of $V\epsilon^{-D}$ points, D being the dimension of the space.

It is quite clear that the map f_ϵ is very far from a RM. In a RM there is nothing like the concept of smoothness. Here if two points i and j are near in physical space (of course their distance cannot be smaller than ϵ) the points $f_\epsilon(i)$ and $f_\epsilon(j)$ will also be correlated. On the contrary, in the RM there will be no correlation whatsoever.

But, as we saw in the previous section, the statistical properties of RM's do not require the strong condition that the successors of different points are independent random variables: it is enough that the closing probabilities become stationary after some time. A condition under which this happens is that the successors of two near configurations become uncorrelated after a sufficiently large number of iterations. This is just what happens in chaotic systems.

Two related heuristic arguments were given in the literature to justify the analogy between discretized chaotic systems and RM [7, 8]; we will now review them, reformulating them in the language of the closing probabilities.

Let us consider the map f_ϵ^l, i.e. the lth iterate of the map f_ϵ. We will argue that this map is essentially a RM on the strange attractor, or better, on the integers which correspond to the points nearest the attractor.

The argument runs as follows:

- We first assume that, as far as the properties of the attractors are concerned, the map f_ϵ^l has the same properties as the map $I(f^l(X(i)))$ obtained from the discretization of the

lth iterate of the exact map. The two integers $f_\epsilon^l(i)$ and $I(f^l(X(i)))$ will not be equal and not even near for large l, but all we need is that the closing probabilities are the same in the two cases.

- We now take $l = l_\epsilon$ large enough so that two random points initially at a distance of order ϵ are completely uncorrelated after l_ϵ iterations. To this purpose, it is enough that l_ϵ is of order $1/\lambda \log \epsilon$, where λ is the largest positive Ljapunov exponent. After such a number of iterations two points initially at a distance of order ϵ will be at a distance of order 1, and the discretization will bring them in completely different boxes. Thus we can argue that the map $I(f^l(X(i)))$ is essentially a random map, in the sense that successors of different points are not correlated. Moreover, if K is large enough the continuous dynamics takes place essentially on the strange attractor.
- The maps f_ϵ and f_ϵ^K have the same attractors with the same period, apart from a possible factor not greater than $K = \mathrm{O}(\ln(\epsilon))$. Since we are interested in the power behaviour as a function of ϵ, a factor proportional to $\ln \epsilon$ is irrelevant.

Let us translate this discussion into the language of closing probabilities. We can define for the continuous map quantities which are equivalent to the closing probabilities of the discretized map. More precisely, let us call $p_\epsilon(t, t+l)$ the probability that two points of the same continuous trajectory are at time steps t and $t+l$ in the same box of size ϵ. Assume that the dynamics is mixing. Then, if t and l are large enough, $p_\epsilon(t, t+l)$ tends to the asymptotic value

$$\pi_\epsilon^* = \sum_i (\mu_i)^2 \tag{20}$$

where μ_i is the invariant measure of the box i. The first hypothesis of the above argument is indeed equivalent to the condition that the closing probabilities $\pi_\epsilon(t, t+l)$ and the functions $p_\epsilon(t, t+l)$ are equal. Thus in the above hypothesis the closing probabilities of the discrete system reach an asymptotic value independent on both l and t, and the properties of generalized random maps hold for it.

This computation does not hold for $t = 0$, since the box where the initial configuration is has to be chosen with uniform probability. In this case thus it holds that $p_M(0, l) = 1/M$, and, following the lines of the computation of the previous section, we find that the number of cycles increases as the logarithm of the time-scale τ_ϵ, where $\tau_\epsilon = (\pi^*)^{-1/2}$.

How is this time-scale related to the discretization constant ϵ? If the attractor is fractal, it holds, asymptotically for $\epsilon \to 0$, $\sum_i (\mu_i)^2 \propto \epsilon^{D_2}$, where D_2 is called the correlation dimension† [22]. Thus the time-scale of the attractors of the disordered dynamical system

† The fractal dimensions of a set $\mathcal{D}$, equipped with a measure $\mathrm{d}\mu(x)$, can be defined in the following way. We divide the space in boxes of size ϵ and we call μ_i the integral of the measure of the ith box. Asymptotically in $\epsilon \to 0$ it holds that

$$\sum_i (\mu_i)^s \propto \epsilon^{D_s} \tag{21}$$

where the exponents D_s are called generalized Renyi dimensions [20].

It is evident that $D_1 = 0$. The total number of boxes which intersect the set diverges as ϵ^{D_0}, with negative D_0, where $-D_0$ is the box-counting dimension. In contrast, the probability of finding two random points in the same box goes to zero as ϵ^{D_2}. D_2 is called the correlation dimension.

If

$$D_s = (s-1)D_H \tag{22}$$

the fractal is homogeneous and D_H is its fractal dimension. In contrast, if D_s is not linear in s the set is said to be multi-fractal [21] and the relations among different dimensions are not trivial.

is related to the fractal dimension of the strange attractor through the equation [7, 8]:

$$\tau_\epsilon \propto \epsilon^{-D_2/2}. \tag{23}$$

On the other hand, the statistical properties of the attraction basins do not depend on the structure of the fractal attractor, as was pointed out in the previous section.

There is another point that we should mention in order to identify a discretized chaotic map with a disordered dynamical system. In the latter, dynamical rules are extracted at random, while the discretization prescription that we gave is deterministic. But this discrepancy is only apparent. For a given lattice spacing ϵ we can choose virtually infinite discretization prescriptions, thus generating different realizations of the discretized dynamical rules with a fixed system size. Two systems with very similar discretization will behave in a very similar way on the small time-scales, but, as the dynamics is chaotic, their evolutions will become uncorrelated on large enough time-scales. In this way we can generate an ensemble of discrete dynamical systems with identical phase space.

In this perspective, the question arises of how the structure of attraction basins change when we change slightly the dynamical rules, either changing the discretization protocol or changing ϵ by a small amount. We will see in section 4 that in this last case numerical simulations suggest the existence of a kind of correlation length for system size: the average basin size appears to change little for a small change of $N = \epsilon^{-1}$, but when the change of N is large enough we see a new value of $\langle Y \rangle$ not more correlated to the previous one.

4. Numerical experiments

4.1. The Henon map

To test the above arguments we simulated the discretized Henon map equation (3) with the standard parameters $a = 1.4$ and $b = 0.3$, for which the dynamics is known to be chaotic. In these numerical experiments all the intermediate computations are done in double precision, but this should not change the results, as long as the boxes are much larger than the rounding error.

In figure 1 we show the average cycle length as a function of ϵ^{-1} in the range from 2.5×10^4 to 2.0×10^6, starting from the point $(1, 0)$. The plot can be fitted to a power of ϵ with exponent 0.62, which coincides within the error with the value $D_2/2 = 0.61$ found in the literature [24]. Also in the same figure, represented as a star, is the result of a single run where the discretization was not imposed in the algorithm but came from the round-off errors in single precision operations ($\epsilon \simeq 2^{-24}$, cycle length 55′574). We made the same for double precision operations ($\epsilon \simeq 2^{-48}$), finding a cycle of length 5.12×10^9 (not represented). Both points agree with the above power law, indicating that there is no difference between discretization and round-off errors generated in the computer.

In figure 2 we show the moments of the distribution of the weights of the attraction basins as function of ϵ^{-1}. Their asymptotic values are in good agreement with the corresponding values in the random map model, as predicted above.

4.2. The logistic map

To investigate what happens in a transition from ordered to chaotic behaviour we simulated the logistic map (2).

In the parameter range where the system is chaotic the predictions of the previous sections seem to be asymptotically verified (but for the values of ϵ that we simulated

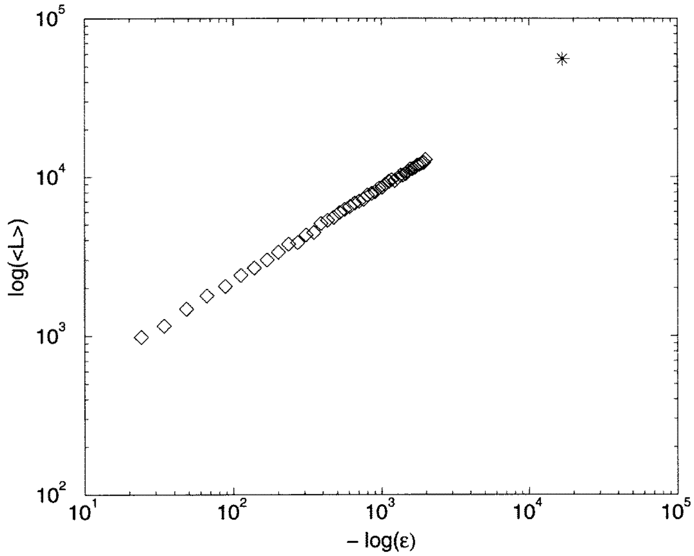

Figure 1. Log–log plot of the average period as a function of ϵ for the Henon map with parameters $a = 1.4$ and $b = 0.3$ (chaotic regime). The best fit exponent, 0.62, coincides up to statistical errors with half of the correlation dimension. The star is the length of the period detected in a single run of the system, using single-precision arithmetic.

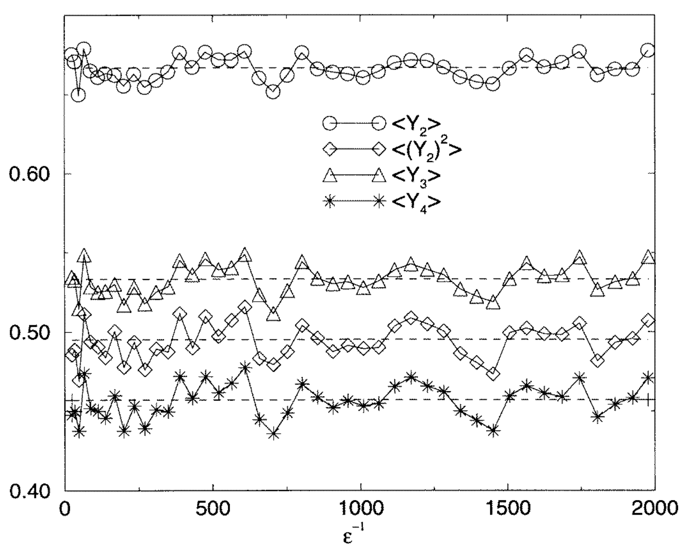

Figure 2. Moments of the weights distribution for the discretized Henon map (parameters: $a = 1.4$, $b = 0.3$) as a function of ϵ^{-1}. The broken lines represent the analytical predictions for the random map.

the corrections are still important). The results for the first moments of attraction basins distribution are showed in figure 3. Figure 3(a) shows the behaviour of $\langle Y_2 \rangle$ as a function of the lattice spacing ϵ. The broken line represents the random map value. Every point is an average over many slightly different values of ϵ, but the discretization procedure is

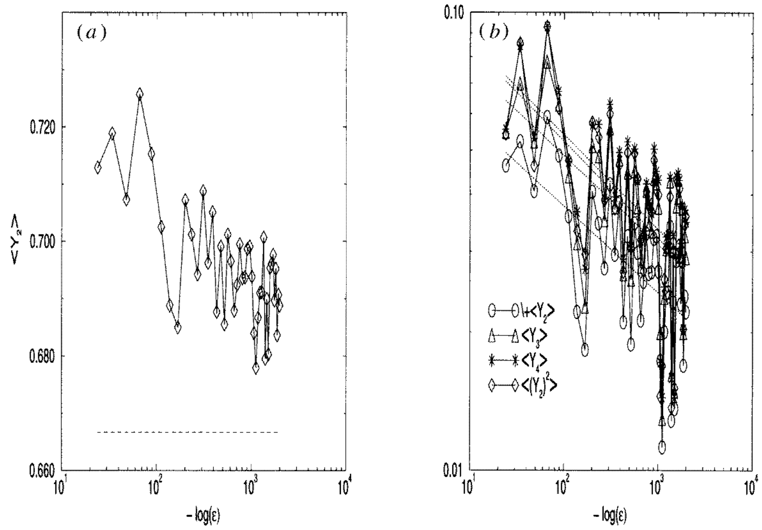

Figure 3. (*a*) Average weight of the attraction basins for the discretized logistic map in the chaotic regime ($a = 2$), as a function of ϵ^{-1} (represented in logarithmic scale). The broken line represents the random map value. (*b*) Log–log plot of the difference between the moments of the attraction basins weight distribution and the analogous random map values. The dotted lines fit the decay with a power law.

always the same. One can notice that there are important correlations between systems with slightly different values of ϵ: when we change $N = \epsilon^{-1}$ by a small amount, the average basin weight changes very little, and only after a large enough change the new value is almost uncorrelated to the old one. The values of the higher moments are strongly correlated to $\langle Y_2 \rangle$. We plot the difference between them and the random map predictions in figure 3(*b*). The corrections to the random map values seem go to zero as a power law of ϵ. The best fit coefficients of the power laws are respectively 0.09 for $\langle Y_2 \rangle$, 0.11 for $\langle Y_3 \rangle$ and 0.14 for $\langle Y_4 \rangle$ and $\langle (Y_2)^2 \rangle$. The exponent is roughly equal to 0.2 for all four curves. Indeed we expect this exponent to be related to the exponent of typical periods, whose value is in this case 0.5. The convergence to the random map distribution is thus slower for a one-dimensional system than for a two-dimensional one, like the Henon map. In this case the numerical data are compatible with our theoretical predictions, but the existence of strong finite-size effects may introduce systematic effects in our extrapolations

When the control parameter a is below a critical threshold, a periodic attractor of length equal to a power of 2 exists and is stable and the dynamics is not chaotic. In this case there should be only one attraction basin, corresponding to the periodic orbit of the continuous system. Nevertheless, finite size effects introduce in the discretized system spurious new attractors, which are just a copy of the 'true' one: consider, for example, the case where an orbit of period 2 is present ($\frac{3}{4} \leqslant a \leqslant 1.238\ldots$). Each of the two points that constitute the real attractor are placed between two points of the discretized system, and in this system the attractors can be two. Thus the average basin weight is less than 1 for finite ϵ, and it goes to 1 as a power law of ϵ: $1 - \langle Y_P \rangle \approx C_P \epsilon^{\gamma_P}$. From our data it appears that the exponent γ_P depends very little on P, showing a small tendency to decrease with P (we cannot say if it is significant: its value is 0.313 for $P = 2$, 0.308 for $P = 3$ and 0.303 for $P = 4$), while the coefficient C_P is a growing function of P (this is quite natural, since the condition $Y_P = 1$ becomes more and more restrictive when P increases).

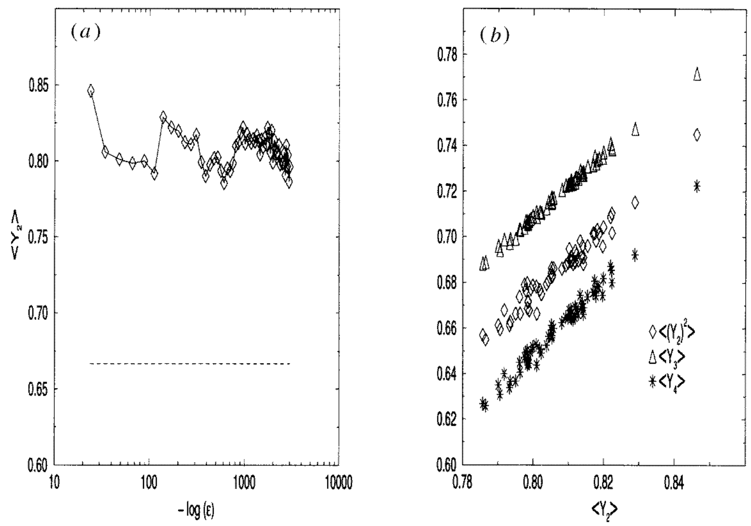

Figure 4. (a) Average weight of the attraction basins for the discretized logistic map at the critical point ($a = 1.401\,1552$), as a function of ϵ^{-1} (represented in logarithmic scale). The broken line represents the random map value. (b) $\langle Y_3 \rangle$, $\langle Y_4 \rangle$ and $\langle (Y_2)^2 \rangle$ versus $\langle Y_2 \rangle$ for the discretized logistic map at the critical point.

At the transition between order and chaos, when the maximum Ljapunov exponent is still zero but periodic orbits disappear, we expect a non-trivial distribution of attraction basin weights, with an average weight intermediate between 1 (periodic orbits) and $\frac{2}{3}$ (random map). This is what is observed: at the critical point $a_c = 1.401\,1552$ the average basin weight seems to go asymptotically to a value larger than for a random map, though the correlations in the sampling that we mentioned above make it difficult to extrapolate the asymptotic value (figure 4(a)). This also holds for the higher moments, which indeed in our sampling are strongly correlated to $\langle Y_2 \rangle$. We plot in figure 4(b) the values of $\langle Y_3 \rangle$, $\langle Y_4 \rangle$ and $\langle Y_2^2 \rangle$ as a function of $\langle Y_2 \rangle$. Our noisy data arrange themselves on more regular curves, which can be fitted to power laws, $\langle Y_p \rangle \approx C_p \langle Y_2 \rangle^{\gamma_p}$. The coefficient C_p is always equal to 1 within the errors (as it should be, since, when Y_2 is equal to 1, all other moments are be equal to 1). The best fit exponents are respectively $\gamma_3 = 1.6$, $\gamma_4 = 1.9$ and $\gamma = 1.7$ concerning $\langle Y_2^2 \rangle$. We notice that, within the error, it holds that $\gamma_p = p/2$. The average cycle length also increases in this case as a power law of ϵ^{-1}, with exponent 0.38 (which in the case of a chaotic map would correspond to a correlation dimension of the strange attractor 0.75).

5. Conclusions

In this paper we have shown that the analogy between the cycles induced by the rounding errors on a chaotic map and the cycles in random maps, proposed some years ago in the literature [7, 8], can be extended to the distribution of the attraction basin weights. Our conjecture implies that the latter should be equal to the corresponding distribution found in the random map model for every map which has a strange attractor. So, for instance, two simulations of a chaotic map starting from different initial configurations should converge to the same limit cycle induced by round-off errors, on the average $\frac{2}{3}$ of the times. Our numerical simulations seem to be in agreement with this result, though finite size effects are large (in particular in the one-dimensional system).

The situation seems to be different at the critical point of parameter space which separates the chaotic and the periodic region. It would be interesting to understand whether the distribution of attraction basin weights at the critical point is universal, i.e. it is the same for all the transitions which belong to the same universality class and are characterized by the same Feigenbaum exponents. If the answer is positive, it would be interesting also to set up an analytic computation of the weight distribution in this case.

Another possible development of this work, which in our opinion deserves attention, is the investigation of the relations between the probability distribution of the distance between configurations and the fractal dimensions. The closing probability that we introduced here as our main tool represents in fact the probability that the distance between the configurations visited at times t and t' (on a trajectory not yet closed at time $t'-1$) is equal to zero. There is thus a relation between the distribution of the distance at $d=0$ in the discretized system and the correlation dimension. This fact is interesting, since one can hope that the distribution of the distance can be investigated analytically, as it can be done for instance in the Kauffman model [23, 11]. The annealed approximation introduced in that context is somehow equivalent to the hypothesis that the distance $d(t, t+l)$ is a Markovian random variable, where t is the time variable. This hypothesis in turn implies, if the hypothesis of the ergodic theorem of random variables are verified (as it seems plausible for the distance in a chaotic map), that the distribution of the distance, and the closing probability in particular, tend to an asymptotic value independent both on t and on l, as was argued in this paper.

The arguments that we presented in this note, although quite reasonable and in agreement with numerical experiments, have only a heuristic value. It would be very interesting to find out if rigorous results can be obtained in this direction.

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References

[1] Ruelle D 1980 Strange attractors *Math. Intelligencer* **2** 126
[2] Schuster H G 1988 *Deterministic Chaos* 2nd edn (Weinheim: VHC)
[3] Ruelle D 1976 *Am. J. Math.* **98** 619
[4] Rannou F 1974 *Astron. Astrophys.* **31** 289
[5] Levy Y E 1982 *Phys. Lett.* **88A** 1
[6] Beck C and Roepstorf G 1987 Effects of phase space discretization on the long time behaviour of dynamical systems *Physica* **25D** 173
[7] Grebogi C, Ott E and Yorke J A 1988 Roundoff-induced periodicity and the correlation dimension of chaotic attractors *Phys. Rev.* A **38** 3688
[8] Beck C 1989 Scaling behaviour of random maps *Phys. Lett.* A **136** 121
[9] Metropolis N and Ulam S 1953 *Am. Math. Monthly* **60** 252
Harris B 1960 *Ann. Math. Stat.* **31** 1045
[10] Derrida B and Flyvbjerg H 1986 The random map model: a disordered model with deterministic dynamics *J. Physique* **48** 971–8
[11] Bastolla U and Parisi G 1996 Closing probabilities in the Kauffman model: an annealed computation *Physica* **98D** 1

[12] Mezard M, Parisi G and Virasoro M A 1988 *Spin Glass Theory and Beyond* (Singapore: World Scientific)
[13] Andreev A V *et al* 1996 *Quantum Chaos, Irreversible Classical Dynamics and Random Matrix Theory* archived at http://xxx.lanl.gov/cond-mat9601001
[14] Marinari E, Parisi G and Ritort F 1994 *J. Phys. A: Math. Gen.* **27** 7645
[15] Parisi G 1994 *Gauge Theory, Spin Glasses and Real Glasses* archived at http://xxx.lanl.gov/cond-mat9411115
[16] Kauffman S A 1969 *J. Theor. Biol.* **22** 437
Kauffman S A 1969 Homoeostasis and differentiation in random genetic control networks *Nature* **244** 177–8
[17] Amit D J 1989 *Modeling Brain Function* (Cambridge: Cambridge University Press)
[18] Parisi G 1993 *J. Stat. Phys.* **72** 857
[19] Bastolla U and Parisi G 1997 Attractors in fully asymmetric neural networks *J. Phys. A: Math. Gen.* to appear
[20] Renyi A 1970 *Probability Theory* (Amsterdam: Elsevier)
[21] Paladin G and Vulpiani A 1987 Anomalous scaling laws in multifractal objects *Phys. Rep.* **156** 147
[22] Grassberger P and Procaccia I 1983 Characterization of strange attractors *Phys. Rev. Lett.* **50** 346
[23] Derrida B and Pomeau Y 1986 Random networks of automata: a simple annealed approximation *Biophys. Lett.* **1** 45–9
[24] Grassberger P 1983 On the fractal dimensions of the Henon attractor *Phys. Lett.* **97A** 227
Grassberger P 1983 *Phys. Lett.* **51A** 135